Pure and Applied Mathematics Journal
2024; 13(5): 72-78
http://www.sciencepublishinggroup.com/j/pamj
doi: 10.11648/j.pamj.20241305.12
ISSN: 2326-9790 (Print); ISSN: 2326-9812 (Online)

A_∞-algebra Structure on Connected Multiplicative Operad

Batkam Mbatchou Vane Jacky III*, **Calvin Tcheka**

Department of Mathematics, University of Dschang, Dschang, Cameroon

Email address:
batkamjacky3@yahoo.com (Batkam Mbatchou Vane Jacky III), calvin.tcheka@univ-dschang.org (Calvin Tcheka)
*Corresponding author

To cite this article:
Batkam Mbatchou Vane Jacky III, Calvin Tcheka. A_∞-algebra Structure on Connected Multiplicative Operad. *Pure and Applied Mathematics Journal, 13*(5), 72-78. https://doi.org/10.11648/j.pamj.20241305.12

Received: 2 August 2024; **Accepted:** 9 September 2024; **Published:** 29 September 2024

Abstract: This work develops the structure of A_∞-algebras on operad theory and also the preservation of this structure by a morphism of operads well defined. This structure defined here is motivated by the important role that play certain particular properties such as multiplication and connectivity on the operads. Another key ingredient used to develop this work is the brace operations; which, combined with the properties cited above allowed to better frame the study of this structure. Thus, this paper show explicitly the existence of an A_∞-algebra structure on any connected multiplicative operad endowed with its brace operations and that this structure is minimal if the operad is only multiplicative. Furthermore, the paper also shows the existence of an operads morphism from an unital associative operad, $\mathcal{A}_{ss}$ to any connected multiplicative operad $\mathcal{O}$ preserving the structure of A_∞-algebras existing on these two operads. And when the operad $\mathcal{O}$ is just multiplicative then there is rather a morphism of operads from the associative operad, $\mathcal{A}_s$ to $\mathcal{O}$ preserving this time the minimal A_∞-algebras structure existing on these operads.

Keywords: A_∞-algebra Structure, Minimal A_∞-algebra Structure, Connected Multiplicative Operad, Brace Operations

1. Introduction

Operads are algebraic devices which encode types of algebras. They are very important in categories with a well notion of homotopy where they are useful for the study of homotopy invariant algebraic structures and hierarchies of higher homotopies. One can already see its trace in the paper of Lazard [15] entitled group laws and analyzers, the basic idea of operads has mainly been developed in Chicago in the seventies by the algebraic topologists (S. MacLane [20], J. Stashef [7], J. P. May [10], J. M. Boardmann, R. Vogt [12], F. R. Cohen [6]) to study loop spaces. Moreover, instead of describing algebras by its generators(operations) and the relatives relations(fundamental identities), one may consider all operations that can be performed on a finite number of variables and the relations between these operations. This structure has been baptized by J. P. May: Operad structure. The main interest of this point of view resides on the fact that one may liken algebras even if they are of different natures.

Furthermore A_∞-algebras (sha algebras = strongly homotopy associative algebras) were invented at the beginning of the sixties by J. Stashef [8] as a tool in the study of *group-like* topological spaces. In the subsequent two decades, A_∞-structures found applications and developments in homotopy theory (see [9, 10, 12]). Their use remains essentially confined to this subject [5, 21]. This changes at the beginning of the nineties when the relevance of A_∞-structures in algebra, geometry and mathematical physics became more and more apparent (cf. e.g. [4, 8, 14]).

This paper uses a connected multiplicative operad from which is defined a face homomorphism. This gives us a chain complex endowed with an appropriate product denoted $\odot$. Using the up mentioned materials, the construction of A_∞-algebra structure on the operad $\mathcal{O}$ yields. This algebraic structure becomes minimal when the operad is only multiplicative. The work ends by constructing an homomorphism of operads $A_{ss} \xrightarrow{f} \mathcal{O}$ (respectively $A_s \xrightarrow{f} \mathcal{O}$) that preserves A_∞-algebra structures(respectively minimal A_∞-algebra structures). Here A_{ss}(respectively A_{ss}) is a unitary associative operad(respectively associative operad) while $\mathcal{O}$ is a given connected multiplicative(respectively multiplicative) operad.

The organization of the paper is as follows: the section 2 gives a general reminders on operads and the last section develops the mains results of the work.

2. General Reminders on Operads

2.1. Conventions and Notation

In the sequel,

1. $\mathbb{K}$ denotes an arbitrary field. Vector spaces, tensor products and linear maps are defined over $\mathbb{K}$ unless otherwise stated.
2. For a given connected multiplicative operad $\mathcal{O}$ with multiplication m and $1_0 \in \mathcal{O}(0)$, let us assume by convention in this work that

$$m \circ_1 1_0 = 1_{\mathcal{O}} = m \circ_2 1_0. \quad (1)$$

2.2. Operad and Right-brace Structure on Operad

In this section, $\mathbb{K}$ is the arbitrary ground field and all the work is in the catgory of vector space $\mathrm{Vect}_{\mathbb{K}}$.

2.2.1. Operad

Operads considered here are over the monoidal category of $\mathbb{K}$-vector spaces. Such operads are said to be symmetric(Σ-operad) if they are equipped with a right action of symmetric groups $\Sigma = \{\Sigma_n, n \in \mathbb{N}\}$ and non symmetric(non Σ-operad) if not. In this section, one can recall the fundamental notions on operads over the category $\mathrm{Vect}_{\mathbb{K}}$ and their related properties. One can refer to [2, 11, 16] for more details.

Definition 2.1. *A symmetric operad(Σ-operad or operad) is a collection of right $\sum$-vector spaces over $\mathbb{K}$, $\{\mathcal{O}(k)|k \geq 1\}$, together with a composition product:*

$$\begin{array}{ccc} \mathcal{O}(k) \otimes \mathcal{O}(n_1) \otimes \cdots \otimes \mathcal{O}(n_k) & \xrightarrow{\gamma^{\mathcal{O}}} & \mathcal{O}(n_1 + \cdots + n_k) \\ x \otimes x_1 \otimes \cdots \otimes x_k & \longmapsto & \gamma^{\mathcal{O}}(x; x_1, \cdots, x_k) \end{array}$$

which satisfies the following axioms:

1. *associativity axiom;*
2. *axiom of unity;*
3. *Σ-equivariant ([2, 11, 16]).*

Definition 2.2. *Consider $\mathcal{O}$ and $\mathcal{O}'$ two Σ-operads with respective composition products $\gamma^{\mathcal{O}}$ and $\gamma^{\mathcal{O}'}$ and respective associated unities $1_{\mathcal{O}}$ and $1_{\mathcal{O}'}$. $\mathcal{O} \xrightarrow{f} \mathcal{O}'$ is called morphism of operads if the collection $\{f_n : \mathcal{O}(n) \longrightarrow \mathcal{O}'(n)\}_{n\geq 0}$ of $\sum$-$\mathbb{K}$-vector space homomorphisms satisfies:*

1. *$f(1_{\mathcal{O}}) = 1_{\mathcal{O}'}$;*
2. *$f_j(\gamma^{\mathcal{O}}(x_0 \otimes x_1 \otimes ... \otimes x_n)) = \gamma^{\mathcal{O}'}(f_n(x_0) \otimes f_{i_1}(x_1) \otimes ... \otimes f_{i_n}(x_n))$ with $j = i_1 + i_2 + ... + i_n$;*
3. *$f_n(x * \sigma) = f_n(x) * \sigma$, with $\gamma(x * \sigma; a_1, \cdots, a_n) = \gamma(x; a_{\sigma^{-1}(1)}, \cdots, a_{\sigma^{-1}(n)})$, for some $x \otimes a_{k_1} \otimes \cdots \otimes a_{k_n} \in \mathcal{O}(n) \otimes \mathcal{O}(k_1) \otimes \cdots \otimes \mathcal{O}(k_n)$ and $\sigma \in \Sigma_n$.*

Remark 2.1.

1. Equivalently a Σ-operad can also be defined by the so called partial composition

$$\begin{array}{cccc} \mathcal{O}(m) \otimes \mathcal{O}(n) & \xrightarrow{\circ_i} & \mathcal{O}(m+n-1) & m \geq i \geq 1 \\ x \otimes y & \longmapsto & x \circ_i y & \end{array}$$

satisfying some properties (see [3] for explicit axioms) The two definitions are related as follows:

$$x \circ_i y = \gamma^{\mathcal{O}}(x; \overbrace{id, \cdots, \underbrace{y}_{i} \cdots id,}^{m-tuple}); m \geq i \geq 1. \quad (2)$$

2. An operad, $\mathcal{O}$, is said to be *multiplicative* if there is an element of degree 2, $m \in \mathcal{O}(2)$ such that $m \circ_1 m = m \circ_2 m$ or $m\{m\} = 0$.
3. An operad $\mathcal{O}$ is said to be *connected* if $\mathcal{O}(0)$ is isomorphic to the ground field $\mathbb{K}$. In the sequel to avoid confusion, denote by $1_0 = 1_{\mathbb{K}}$ the unit in $\mathcal{O}(0)$
4. Consider a connected operad $\mathcal{O}$. It has been shown in [3] that for all $S \subset [n] = \{1, 2, \cdots, n\}$, with cardinality $l < n$, $\mathcal{O}$ is endowed with a degeneracy map $|_S$ defined as follows:

$$\begin{array}{ccc} |_S\colon \mathcal{O}(n) & \longrightarrow & \mathcal{O}(l) \\ p & \mapsto & p\,|_S = p(x_1, ..., x_n), \end{array}$$

where $x_i = \begin{cases} 1_1 & \text{if} \quad i \in S \\ 1_0 & \text{if not} \end{cases}$

In particular for $n \in \mathbb{N} - \{0\}$ and $1 \leq i \leq n$, set $S_i = [n]-\{i\} = \{1, 2, \cdots, i-1, \hat{i}, i+1, \cdots, n\}$, where $\hat{i}$ means that the natural number i has been omitted.

The face map is defined for $n \geq 2$ as follows:

$$\begin{array}{ccc} F_i^n = |_{S_i}\colon \mathcal{O}(n) & \longrightarrow & \mathcal{O}(n-1) \\ p & \mapsto & F_i^n(p) = p\,|_{S_i} = p \circ_i 1_0 \end{array}$$

and $F^1 = |_{\emptyset}\colon \mathcal{O}(1) \longrightarrow \mathcal{O}(0)$ such that for $1_{\mathcal{O}} \in \mathcal{O}(1)$, $F^1(1_{\mathcal{O}}) = 1_{\mathcal{O}}\,|_{\emptyset} = 1_0 \in \mathcal{O}(0)$.

One can verify easily by straightforward computation that the $\mathbb{K}$-linear map F_i^n is subject to the following properties:

$$F_i F_j = F_{j-1} F_i \quad \text{if} \quad i < j \quad (3)$$

$$F_i F_j = F_j F_{i+1} \quad \text{if} \quad i \geq j \quad (4)$$

(See [3] for more detail).

The fundamental example is the endomorphism operad denoted here by $\mathcal{L}_A := End_A$, for an object A in the category of $\mathbb{K}$-vector spaces.

Examples 2.1. For a given associative unitary $\mathbb{K}$-algebra A with multiplication μ_A and unit η_A, the operad $\mathcal{L}_A$ of multilinear homomorphisms defined by: for all $n \geq 1$, $\mathcal{L}_A(n) := Hom_{\mathbb{K}}(A^{\otimes n}, A)$, is a multiplicative operad.

1. The associated operadic composition $\gamma^{\mathcal{L}_A}$ is substitution of the values of n operations in a n-ary operation as inputs.
2. Its associated multiplication is $m := \mu_A \in \mathcal{L}_A(2)$.
3. Its associated unit element is the identity map $A \xrightarrow{id_A} A$.

2.2.2. Right-brace Structure on Operad (See [3])

Let $\mathcal{O} = \bigoplus_{k\geq 0} \mathcal{O}(k)$, be the sum of all components of a connected multiplicative operad in the category of the $\mathbb{K}$-

vector spaces, $\mathcal{E}_{\mathbb{K}-vect}$.

Let us set

$$(\mathcal{O}^{\otimes n})(s) = \bigoplus_{s_1+s_2+\cdots+s_n=s} \mathcal{O}(s_1)\otimes\mathcal{O}(s_2)\otimes\cdots\otimes\mathcal{O}(s_n) \quad (5)$$

for all $s \in \mathbb{N}$,. For $p \in \mathcal{O}(r)$, the degree of p is the integer r and it is denoted by $deg(p) = r$ and $|p| = r - 1$ denotes the degree of its suspension.

Definition 2.3. A right brace operations on any operad $\mathcal{O}$ is the collection of multilinear operations defined by:

$$\begin{gathered} \mathcal{O} \otimes \mathcal{O}^{\otimes n} \longrightarrow \mathcal{O}, n \geq 1 \\ p \otimes (q_1 \otimes q_2 \otimes \cdots \otimes q_n) \mapsto p\{q_1, q_2, \cdots, q_n\} \end{gathered}$$

such that for $p, q_1, q_2, \cdots, q_n \in \mathcal{O}$,

$$\begin{aligned} p\{q_1, q_2, \cdots, q_n\} = \sum (-1)^{\epsilon} \gamma(p; 1_{\mathcal{O}}, ..., 1_{\mathcal{O}}, q_1, 1_{\mathcal{O}}, ..., 1_{\mathcal{O}}, \\ q_2, 1_{\mathcal{O}}, ..., 1_{\mathcal{O}}, q_n, 1_{\mathcal{O}}, ..., 1_{\mathcal{O}}), \end{aligned} \quad (6)$$

where the sum runs over all possible substitutions of $q_1, q_2, \cdots, q_n$ into p in the prescribed order and

$$\epsilon := \sum_{j=1}^{n} \mid q_j \mid (deg(p\{q_1, \cdots, q_n\}) - i_j)), \quad (7)$$

i_j being the total number of inputs in front of q_j.

Thus the right braces $p\{q_1, q_2, \cdots, q_n\}$ are homogeneous of degree $-n$, i.e.;

$$deg(p\{q_1, q_2, \cdots, q_n\}) = deg(p) + \sum_{j=1}^{n} \mid q_j \mid \quad (8)$$

with the following conventions: $p \circ q := p\{q\}$ and $p\{\} := p$, for all $p, q \in \mathcal{O}$.

Remark 2.2. Similarly, one can also define the sign of the right-brace as follows

$$\varepsilon' := \sum_{j=1}^{n} \mid q_j \mid t_j \quad (9)$$

where t_j is the total number of inputs after q_j without count the inputs of q_j.

It is clear that the result is obtained easily:

ε and ε' are even (or odd) in the same moment;

that is to say

$$(-1)^{\varepsilon} = (-1)^{\varepsilon'}.$$

See [3] for the following observations to highlight the differences and analogies between the above defined right-brace structure and the one given by Grestenhaber-Voronov.

Definition 2.4. Let $\mathcal{O}$ be an operad with multiplication $m \in \mathcal{O}(2)$ over the category $\mathcal{E}_{\mathbb{K}-vect}$. The odot-product on $\mathcal{O}$ is a linear map denoted $\odot$ and defined by:

$$\begin{array}{ccccc} \mathcal{O} \otimes \mathcal{O} & \xrightarrow{\odot} & \mathcal{O} & & \\ p \otimes q & \mapsto & p \odot q & := & m(p,q) = (m \circ_2 q) \circ_1 p \\ & & & = & \gamma(m; p, q) \\ & & & = & (-1)^{s(r-1)} m\{p, q\} \end{array}$$

where γ is a composition product on $\mathcal{O}$, $p \in \mathcal{O}(r)$ and $q \in \mathcal{O}(s)$.

Proposition 2.1. Consider a connected multiplicative operad $\mathcal{O}$ with multiplication, $m \in \mathcal{O}(2)$, endowed with a right brace structure over the category $\mathcal{E}_{\mathbb{K}-vect}$. Then $(\mathcal{O}, \odot, \partial)$ is a differential graded algebra.

2.2.3. Derived Operator (See [22] for More Details)

Definition 2.5. (derivative of the face operator) Let $n \in \mathbb{N}^*$ and $1 \leq i \leq n$. The operator $(F_i^n)' = F_{i+1}^{n+1}$ is called derivative of the ith face morphism F_i^n.

Remark 2.3. (see [22] for more details)

Let $\mathcal{O}$ be a connected multiplicative operad, $p > q$ two non-negative integer and $f : \mathcal{O}(p) \longrightarrow \mathcal{O}(q)$ an operator.

Obseve that $f = F_{t_1} \circ F_{t_2} \circ \cdots \circ F_{t_{p-q}}, 1 \leq t_i \leq p, 1 \leq i \leq p - q$ and the associated derived operator denoted f' is defined as follows:

$$f' : \mathcal{O}(p+1) \longrightarrow \mathcal{O}(q+1)$$

such that

$$f' = F'_{t_1} \circ F'_{t_2} \circ \cdots \circ F'_{t_{p-q}}, 1 \leq t_i \leq p, 1 \leq i \leq p - q. \quad (10)$$

Examples 2.2.

$$\partial_n = \sum_{i=1}^{n} (-1)^i F_i^n : \mathcal{O}(n) \longrightarrow \mathcal{O}(n-1), \quad (11)$$

then

$$\begin{aligned} \partial'_n &= (\sum_{i=1}^{n} (-1)^i F_i^n)' \\ &= (F_1^n)' + (F_2^n)' + ... + (-1)^n (F_{n-1}^n)' \\ &= \sum_{i=2}^{n+1} (-1)^{i-1} F_i^{n+1}. \end{aligned} \quad (12)$$

Moreover $\partial' x_1 = 0$ and $\partial' x_0$ is not defined. Since ∂'_n is defined for $n \geq 2$ (with $\partial' x_1 = 0$) then it is convenient to modify the definition in degree 1. Let us set

$$\begin{cases} \partial_n^* x_1 = 0 & \text{if } n = 1 \\ \partial_n^* x_n = \sum_{i=2}^{n} (-1)^{i-1} F_i^n x_n & \text{if } n \geq 2 \end{cases} \quad (13)$$

Thus

$$\partial x_n = -F_1 x_n - \partial^* x_n, \;\; x_n \in \mathcal{O}(n) \quad (14)$$

with $\partial^* \partial^* = 0 = \partial' \partial'$ in all degree.

Proposition 2.2. Consider a connected multiplicative operad $\mathcal{O}$. Then $(\mathcal{O}, \odot, \bar{\partial^\star})$ is a differential graded algebra where $\bar{\partial^\star} = -\partial^\star$.

3. Main Results

This section introduce by using somes properties the $\mathcal{A}_\infty$-algebra structure in the framework of operads. This structure will allow us to extend some well-known results in certain particular cases for instance on the unital associative operad, $\mathcal{A}_{ss}$ and on the associative operad $\mathcal{A}_s$).

3.1. Recall on A_∞-algebras Structures

Definition 3.1. An A_∞-algebra over a field $\mathbb{K}$ (also called a 'strongly homotopy associative algebra' or an 'sha algebra') is a $\mathbb{Z}$-graded vector space

$$A = \bigoplus_{p\in\mathbb{Z}} A^p$$

equipped with graded maps (homogeneous $\mathbb{K}$-linear maps)

$$m_n : A^{\otimes n} \longrightarrow A,\ n \geq 1,$$

of degree n-2 such that the following relations work:

1. $m_1 m_1 = 0$, i.e. (A, m_1) is a complex.
2. $m_1 m_2 = m_2(m_1 \otimes 1 + 1 \otimes m_1)$ as maps $A^{\otimes 2} \longrightarrow A$. Here 1 denotes the identity map of the space A. So m_1 is a (graded) derivation with respect to the multiplication m_2.
3. $A^{\otimes 3} \xrightarrow{m_3} A$ is a $\mathbb{K}$-linear map satisfying $m_2(1 \otimes m_2 - m_2 \otimes 1) = m_1 m_3 + m_3(m_1 \otimes 1 \otimes 1 + 1 \otimes m_1 \otimes 1 + 1 \otimes 1 \otimes m_1)$. Observe that the left hand side is the associator for m_2 and that the right hand side can be viewed as the boundary of m_3 in the morphism complex $Hom_{\mathbb{K}}(A^{\otimes 3}, A)$. This leads that m_2 is associative up to homotopy.
4. For $n \geq 1$, we have in general way

$$\sum(-1)^{r+st} m_u(1^{\otimes r} \otimes m_s \otimes 1^{\otimes t}) = 0 \qquad (15)$$

with the above summation running over all decompositions n=r+s+t and we put u=r+1 +t.

Remark 3.1. Observe that when these formulas are applied to elements, additional signs appear because of the Koszul sign rule: for instance,

$$(m_1 \otimes 1 + 1 \otimes m_1)(x \otimes y) = m_1(x) \otimes y + (-1)^{|x|} x \otimes m_1(y). \qquad (16)$$

Therefore $m_1 \otimes 1 + 1 \otimes m_1$ is the usual differential on the tensor product.

Hereafter are given some immediate results of the above definition:

(1) An A_∞-algebra A is not associative but its homology $H^*A = H^*(A, m_1)$ is an associative graded algebra for the multiplication induced by m_2^A.

(2) If $A_p = 0$ for all $p \neq 0$, then $A = A^0$ is an ordinary associative algebra. Indeed, since m_n is of degree n-2, all m_n other than m_2 vanish.

(3) If m_n vanishes for all $n \geq 3$, then A is an associative differential $\mathbb{Z}$-graded algebra and conversely each associative differential $\mathbb{Z}$-graded algebra yields an A_∞-algebra with $m_n = 0$ for all $n \geq 3$.

Definition 3.2. An A_∞-algebra is said to be minimal if its differential m_1 vanishes.

Theorem 3.1. (Kadeishvili [24], see also [1, 13, 17, 18, 19, 23]). For all A_∞-algebra A, the homology H^*A has an A_∞-algebra structure such that

1) $m_1 = 0$ and m_2 is induced by m_2^A, the product on A.
2) there exists a quasi-isomorphism of A_∞-algebras $H^*A \to A$ lifting the identity of H^*A.

3.2. A_∞-algebra Structure on Connected Multiplicative Operad

Unless otherwise stated, operads in this section are equipped with its right-brace structure and any operad will be connected multiplicative.

Theorem 3.2. Consider a connected multiplicative operad $\mathcal{O}$ with multiplication m. There is an A_∞-algebra structure on $\mathcal{O}$ given by the set of morphisms $\{m_n\}_{n\geq 1}$ of degree n-2 defined as follows

1. m_1 is the boundary operator $\bar{\partial}^\star$.

$$m_1 : \mathcal{O} \to \mathcal{O}$$

$$x \mapsto m_1 x = \bar{\partial}^* x = -\sum_{i=2}^{n} (-1)^{i-1} F_i^n x$$

2. m_2 is the product $\odot$ that is to say

$$m_2 : \mathcal{O} \otimes \mathcal{O} \longrightarrow \mathcal{O}$$

such that

$$\begin{aligned} m_2(x \otimes y) &= x \odot y \\ &= (-1)^{|y| deg x} m\{x, y\} \\ &= \gamma(m; x, y) \\ &= m(x, y). \end{aligned}$$

3. Since m is the multiplication of the operad $\mathcal{O}$ i.e., $m \circ_1 m = m \circ_2 m$, then define the morphism m_3 as follows: for $x \in \mathcal{O}(p), y \in \mathcal{O}(q)$ and $z \in \mathcal{O}(r)$,
$$m_3 : \mathcal{O} \otimes \mathcal{O} \otimes \mathcal{O} \longrightarrow \mathcal{O}$$
$$x \otimes y \otimes z \mapsto m_3(x \otimes y \otimes z) = (-1)^{(p+q-1)r}(m \circ_1 m)\{x \odot y, z\} = (-1)^{(p+q-1)r}(m \circ_1 m)\{m_2(x \otimes y), z\}$$
4. The generalization of the definition of m_n for $n \geq 4$ is as follows: for all $x_i \in \mathcal{O}(p_i)$, $1 \leq i \leq n$,
$$m_n(x_1 \otimes \cdots \otimes x_n) = (-1)^{|m_{n-1}(x_1 \otimes \cdots \otimes x_{n-1})| p_n}(m \circ_1 m)\{m_{n-1}(x_1 \otimes \cdots \otimes x_{n-1}), x_n\}.$$

Moreover, if the operad $\mathcal{O}$ is not connected, one may set $m_1 = 0$ and the structure of A_∞-algebra on $\mathcal{O}$ becomes minimal, that is, any multiplicative operad has a minimal A_∞-algebra structure.

Proof It just suffices to show that the sequence $\{m_n\}_{n\geq 1}$ of morphism with $deg(m_n) = n - 2$ satisfy the following conditions

(a) $m_1 m_1 = 0$
(b) $m_1 m_2 = m_2(m_1 \otimes 1 + 1 \otimes m_1)$
(c) $m_2(1 \otimes m_2 - m_2 \otimes 1) = m_1 m_3 + m_3(m_1 \otimes 1 \otimes 1 + 1 \otimes m_1 \otimes 1 + 1 \otimes 1 \otimes m_1)$
(d) More generally $\sum (-1)^{r+st} m_u(1^{\otimes r} \otimes m_s \otimes 1^{\otimes t}) = 0$, for $n \geq 4$, where the sum runs over all decompositions n=r+s+t and we put u=r+1+t.

Since $(\mathcal{O}, \odot = m_2, \bar{\partial}^\star = m_1)$ is a differential graded algebra, conditions (a) and (b) are satisfied.

Now, let us show condition (c).

Consider $x \in \mathcal{O}(p), y \in \mathcal{O}(q), z \in \mathcal{O}(r)$. Since m is the multiplication of operad one can have

$$[m_2(1 \otimes m_2 - m_2 \otimes 1)](x \otimes y \otimes z) = m_2(x \otimes m_2(y \otimes z)) - m_2(m_2(x \otimes y) \otimes z) = m(x, m(y,z)) - m(m(x,y),z)$$
$$= (m \circ_2 m)(x \otimes y \otimes z) - (m \circ_1 m)(x \otimes y \otimes z) = (m \circ_2 m - m \circ_1 m)(x \otimes y \otimes z) = 0.$$

Since $m_2(1 \otimes m_2 - m_2 \otimes 1) = 0$, it remains to show that $m_3(m_1 \otimes 1 \otimes 1 + 1 \otimes m_1 \otimes 1 + 1 \otimes 1 \otimes m_1) + m_1 m_3 = 0$. Thus, one can have:

1. computation of $m_3(m_1 \otimes 1 \otimes 1 + 1 \otimes m_1 \otimes 1 + 1 \otimes 1 \otimes m_1)$

$$[m_3(m_1 \otimes 1 \otimes 1 + 1 \otimes m_1 \otimes 1 + 1 \otimes 1 \otimes m_1)](x \otimes y \otimes z) m_3[m_1(x) \otimes y \otimes z + (-1)^p x \otimes m_1(y) \otimes z + (-1)^{p+q} x \otimes y \otimes m_1(z)]$$
$$= \underbrace{m_3(\bar{\partial}^* x \otimes y \otimes z)}_{(1)} + \underbrace{(-1)^p m_3(x \otimes \bar{\partial}^* y \otimes z)}_{(2)} + \underbrace{(-1)^{p+q} m_3(x \otimes y \otimes \bar{\partial}^* z)}_{(3)}.$$

Step by step, let us compute (1), (2) and (3).

$$(3) = (-1)^{p+q} m_3(x \otimes y \otimes \bar{\partial}^* z) = (-1)^{p+q}(-1)^{(p+q-1)(r-1)}(m \circ_1 m)\{x \odot, z\}$$
$$= (-1)^{(p+q-1)r+1}(m \circ_1 m)\{m(x,y), z\}$$
$$= \underbrace{(-1)^{r+1} m(m(m(x,y), \bar{\partial}^* z), 1_{\mathcal{O}})}_{(a)} - \underbrace{m(m(m(x,y), 1_{\mathcal{O}}), \bar{\partial}^* z)}_{(b)} + \underbrace{(-1)^{p+q} m(m(1_{\mathcal{O}}, m(x,y)), \bar{\partial}^* z)}_{(c)}.$$

$$(2) = (-1)^p m_3(x \otimes \bar{\partial}^* y \otimes z) = (-1)^{p+(p+q)r}(m \circ_1 m)\{x \odot \bar{\partial}^* y, z\}$$
$$= (-1)^{p+(p+q)r}(m \circ_1 m)\{m(x, \bar{\partial}^* y), z\}$$
$$= (-1)^{p+(p+q)r}[(-1)^{(p+q)(r+1)+(r-1)} m(m(m(x, \bar{\partial}^* y), z), 1_{\mathcal{O}})$$
$$+ (-1)^{(p+q-2)(r+1)} m(m(m(x, \bar{\partial}^* y), 1_{\mathcal{O}}), z) + (-1)^{(p+q-2)r} m(m(1_{\mathcal{O}}, m(x, \bar{\partial}^* y)), z)]$$
$$= \underbrace{(-1)^{q+r-1} m(m(m(x, \bar{\partial}^* y), z), 1_{\mathcal{O}})}_{(d)} + \underbrace{(-1)^q m(m(m(x, \bar{\partial}^* y), 1_{\mathcal{O}}), z)}_{(e)} + \underbrace{(-1)^p m(m(1_{\mathcal{O}}, m(x, \bar{\partial}^* y)), z)}_{(f)}.$$

$$(1) = m_3(\bar{\partial}^* x \otimes y \otimes z) = (-1)^{(p+q-2)r}(m \circ_1 m)\{\bar{\partial}^* x \odot y, z\}$$
$$= (-1)^{(p+q)r}(m \circ_1 m)\{m(\bar{\partial}^* x, y), z\} = (-1)^{(p+q)r}[(-1)^{(p+q-2)(r+1)+(r-1)} m(m(m(\bar{\partial}^* x, y), z), 1_{\mathcal{O}})$$
$$+ (-1)^{(p+q-2)(r+1)} m(m(m(\bar{\partial}^* x, y), 1_{\mathcal{O}}), z) + (-1)^{(p+q-2)r} m(m(1_{\mathcal{O}}, m(\bar{\partial}^* x, y)), z)]$$
$$= \underbrace{(-1)^{q+q+r-1} m(m(m(\bar{\partial}^* x, y), z), 1_{\mathcal{O}})}_{(g)} + \underbrace{(-1)^{p+q} m(m(m(\bar{\partial}^* x, y), 1_{\mathcal{O}}), z)}_{(h)} + \underbrace{m(m(1_{\mathcal{O}}, m(\bar{\partial}^* x, y)), z)}_{(i)}.$$

2. Hereafter the computation of the last term $m_1 m_3$

$$(m_1 m_3)(x \otimes y \otimes z) = (-1)^{(p+q-1)r} m_1[(m \circ_1 m)\{x \odot y, z\}]$$
$$= (-1)^{(p+q-1)r} m_1[(m \circ_1 m)\{m(x,y), z\}] = (-1)^{(p+q-1)r} m_1[(-1)^{(p+q)(r+1)} m(m(m(x,y), z), 1_{\mathcal{O}}) +$$
$$(-1)^{(p+q-1)(r+1)} m(m(m(x,y), 1_{\mathcal{O}}), z) + (-1)^{(p+q-1)r} m(m(1_{\mathcal{O}}, m(x,y)), z)]$$
$$= \underbrace{(-1)^{q+q+r} \bar{\partial}^*[m(m(m(x,y), z), 1_{\mathcal{O}})]}_{(1')} + \underbrace{(-1)^{p+q-1} \bar{\partial}^*[m(m(m(x,y), 1_{\mathcal{O}}), z)]}_{(2')} + \underbrace{\bar{\partial}^*[m(m(1_{\mathcal{O}}, m(x,y)), z)]}_{(3')}.$$

Step by step, one may compute (1'), (2') and (3')

$$(1') = (-1)^{q+q+r} \bar{\partial}^*[m(m(m(x,y), z), 1_{\mathcal{O}})]$$
$$= (-1)^{q+q+r}[m(m(\bar{\partial}^* m(x,y), z), 1_{\mathcal{O}}) + (-1)^{p+q} m(m(m(x,y), \bar{\partial}^* z), 1_{\mathcal{O}}) + (-1)^{p+q+r} m(m(m(x,y), z), \underbrace{\bar{\partial}^* 1_{\mathcal{O}}}_{=0})]$$
$$= \underbrace{(-1)^{p+q+r} m(m(m(\bar{\partial}^* x, y), z), 1_{\mathcal{O}})}_{(g')} + \underbrace{(-1)^{q+r} m(m(m(x, \bar{\partial}^* y), z), 1_{\mathcal{O}})}_{(d')} + \underbrace{(-1)^r m(m(m(x,y), \bar{\partial}^* z), 1_{\mathcal{O}})}_{(a')}.$$

$$(2') = (-1)^{p+q-1} \bar{\partial}^*[m(m(m(x,y), 1_{\mathcal{O}}), z)]$$
$$= \underbrace{(-1)^{p+q-1} m(m(m(\bar{\partial}^* x, y), 1_{\mathcal{O}}), z)}_{(h')} + \underbrace{(-1)^{q-1} m(m(m(x, \bar{\partial}^* y), 1_{\mathcal{O}}), z)}_{(e')} + \underbrace{-m(m(m(x,y), 1_{\mathcal{O}}), \bar{\partial}^* z)}_{(b')}.$$

$$(3') = \bar{\partial}^*[m(m(1_{\mathcal{O}}, m(x,y)), z)]$$
$$= \underbrace{-m(m(1_{\mathcal{O}}, m(\bar{\partial}^* x, y)), z)}_{(i')} + \underbrace{(-1)^{p+1} m(m(1_{\mathcal{O}}, m(x, \bar{\partial}^* y)), z)}_{(f')} + \underbrace{(-1)^{p+q+1} m(m(1_{\mathcal{O}}, m(x,y)), \bar{\partial}^* z)}_{(c')}.$$

Comparing the two results obtained in 1 and 2, notice that (a)+(a')=0 until (i)+(i')=0 so the equality

$$m_2(1 \otimes m_2 - m_2 \otimes 1) = m_1 m_3 + m_3(m_1 \otimes 1 \otimes 1 + 1 \otimes m_1 \otimes 1 + 1 \otimes 1 \otimes m_1)$$

works.

Moreover, by using the associativity of the right brace and the one of the multiplication m of operad, it is clearly easy to show that $\sum(-1)^{r+st}m_u(1^{\otimes r}\otimes m_s\otimes 1^{\otimes t}) = 0$, for $n = r+s+t \geq 4$.

Examples 3.1. There are too many examples of operad with an A_∞-algebra structure for instance:

1. the unital associative operad, $\mathcal{A}_{ss} = \{\mathbb{K}[S_n], n \geq 0\}$ has a A_∞-algebra structure.
2. For a given associative $\mathbb{K}$-algebra A, the reduced endomorphism operad $\bar{\mathcal{L}}_A = \{\bar{\mathcal{L}}_A(n)\}_{n\geq 0}$ defined as follows

$$\bar{\mathcal{L}}_A(n) = Hom_{\mathbb{K}}(A^{\otimes n}, A),\ if\ n \geq 1$$

$$\bar{\mathcal{L}}_A(0) = \mathbb{K}$$

   has a A_∞-algebra structure.
3. The associative operad, $\mathcal{A}_s = \{\mathbb{K}[S_n], n \geq 1\}$ has a minimal A_∞-algebra structure.
4. For a given associative $\mathbb{K}$-algebra A, the endomorphism operad

$$\mathcal{L}_A = \{\mathcal{L}_A(n)\}_{n\geq 0} = \{Hom_{\mathbb{K}}(A^{\otimes n}, A)\}_{n\geq 0}$$

   has a minimal A_∞-algebra structure.

Proposition 3.1. Consider a connected multiplicative operad $\mathcal{O}$. There is a morphism of operads $f : \mathcal{A}_{ss} \longrightarrow \mathcal{O}$ which preserves the A_∞-algebras structures. Moreover the operad $\mathcal{O}$ is just multiplicative, then There is a morphism of operads $f : \mathcal{A}_s \longrightarrow \mathcal{O}$ which preserves the minimal A_∞-algebras structures.

Proof If the operad $\mathcal{O}$ is connected multiplicative then there is an operads morphism $f : \mathcal{A}_{ss} \to \mathcal{O}$ defined as follows:

$$f_0 = \mathrm{Id} : \mathbb{K} \to \mathcal{K}$$
$$f_1 = \eta : \mathbb{K} \to \mathcal{O}(1), \text{ the unit map of operad}$$
$$f_2 : \mathbb{K}[\textstyle\sum_2] \to \mathcal{O}(2); \textstyle\sum_2 = \{id = (12), \tau = (21)\}$$
$$\sigma \mapsto \begin{cases} m & \text{if } \sigma = id \\ \tau m & \text{if } \sigma = \tau \end{cases}$$

If $n \geq 3$, then f_n is induced by f_2 and the fact that m is associative. For instance if n=3, one have $f_3 : \sigma \in \sum_3 \mapsto \sigma\gamma_2^{\mathcal{O}}(f_2 \otimes \eta \otimes f_2)$. Thus, for $n \geq 4$

$$f_n = \gamma_2^{\mathcal{O}}(f_2 \otimes \eta \otimes f_{n-1}).$$

For all $n \in \mathbb{N}^*$, f_n is a map in $Lin_{\mathbb{K}} = \mathcal{E}_{vect}$ and f preserves the unit (because $f_1(1_{\mathbb{K}}) = \eta(1_{\mathbb{K}})$ which is the unit element of operad $\mathcal{O}$). By definition f is compatible with the composition products on $\mathcal{A}_{ss}$ and $\mathcal{O}$. Therefore $f = \{f_n\}_{n\geq 1} : \mathcal{A}_{ss} \longrightarrow \mathcal{O}$ is a morphism of operads.

So it suffices to show the compatibility of that morphism with the structures of (minimal) A_∞-algebras on $\mathcal{A}_{ss}$ (or $\mathcal{A}_s$) and $\mathcal{O}$ i.e., the following equation

$$m_n^{\mathcal{O}} \circ (f_{t_1} \otimes \cdots \otimes f_{t_n}) = f_u \circ m_n$$

works, with $u = t_1 + \cdots t_n + n - 2$. Let $\sigma_i \in \mathcal{A}_{ss}(t_i), 1 \leq i \leq n$ then

$$[m_n^{\mathcal{O}} \circ (f_{t_1} \otimes \cdots \otimes f_{t_n})](\sigma_1 \otimes \cdots \otimes \sigma_n) = m_n^{\mathcal{O}}(f_{t_1}(\sigma_1) \otimes \cdots \otimes f_{t_n}(\sigma_n))$$
$$=(-1)^{|m_{n-1}^{\mathcal{O}}(f_{t_1}(\sigma_1)\otimes\cdots\otimes f_{t_{n-1}}(\sigma_{n-1}))|t_n}(m \circ_1 m)\{m_{n-1}(f_{t_1}(\sigma_1) \otimes \cdots \otimes f_{t_{n-1}}(\sigma_{n-1})), f_{t_n}(\sigma_n)\}.$$

Moreover, for the multiplication $\tau = (12)$ of operad $\mathcal{A}_{ss}$, note that

$$(f_u \circ m_n)(\sigma_1 \otimes \cdots \otimes \sigma_n) = (-1)^{|m_{n-1}(\sigma_1\otimes\cdots\otimes\sigma_{n-1})|t_n} f_u[(\tau \circ_1 \tau)\{m_{n-1}(\sigma_1 \otimes \cdots \otimes \sigma_{n-1}), \sigma_n\}]$$
$$=(-1)^{|m_{n-1}(\sigma_1\otimes\cdots\otimes\sigma_{n-1})|t_n} f_3((\tau \circ_1 \tau))\{f_{u-t_n-3}(m_{n-1}(\sigma_1 \otimes \cdots \otimes \sigma_{n-1})), f_{t_n}(\sigma_n), \text{[since f morphism of operads i.e.,}$$
$$f(\gamma(x; y_1, \cdots, y_n)) = \gamma'(f_n(x), f_{t_1}(y_1), \cdots, f_{t_n}(y_n))]$$
$$=(-1)^{|m_{n-1}(\sigma_1\otimes\cdots\otimes\sigma_{n-1})|t_n}(m \circ_1 m)\{m_{n-1}(f_{t_1}(\sigma_1) \otimes \cdots \otimes f_{t_{n-1}}(\sigma_{n-1})), f_{t_n}(\sigma_n)\},$$
$$[\text{since } f_3((\tau \circ_1 \tau)) = f_2(\tau) \circ_1 f_2(\tau) = m \circ_1 m]$$
$$=[m_n^{\mathcal{O}} \circ (f_{t_1} \otimes \cdots \otimes f_{t_n})](\sigma_1 \otimes \cdots \otimes \sigma_n).$$

Thus, by using recurssive definition of m_{n-1} and the morphism of operads f it is easy to have the result.

4. Conclusion

This paper was about studying the structure of A_∞-algebras on operads. In order to achieve this, the properties of multiplication and connectivity on the operads and the definition of brace operations played a very important role. This structure defined on the operads extends or add the new algebraic structure in certain particular cases such as for example the operads given in examples 3.1.

Conflicts of Interest

The authors declare no conflicts of interest.

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